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The local Lehmer inequality for Drinfeld modules

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Abstract

We give a lower bound for the local height of a nontorsion element of a Drinfeld module.

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1. Introduction

The classical Lehmer conjecture (see [11, p. 476]) asserts that there is an absolute constant $C > 0$ so that any algebraic number α that is not a root of unity satisfies the following inequality for its logarithmic height

$$h(\alpha) \geqslant \frac{C}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}.$$

The best known general result towards this conjecture is obtained in [4] (see also Section 2.1 of [15] for a review of all results known so far towards Lehmer conjecture). The analog of Lehmer conjecture for elliptic curves and abelian varieties has also been much studied (see [1,2,10,12, 15]). The paper [3] formulated a conjecture whose general form is Conjecture 1.1, which we refer to as the Lehmer inequality for Drinfeld modules.

Our notation for Drinfeld modules follows the one from [9]: p is a prime number and q is a power of p. We denote by $\mathbb{F}_q$ the finite field with q elements. We let C be a nonsingular

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projective curve defined over $\mathbb{F}_q$ and we fix a closed point ∞ on C. Then we define A as the ring of functions on C that are regular everywhere except possibly at ∞.

We let K be a field extension of $\mathbb{F}_q$. We fix a morphism $i: A \to K$. We define the operator τ as the power of the usual Frobenius with the property that for every x, $\tau(x) = x^q$. Then we let $K\{\tau\}$ be the ring of polynomials in τ with coefficients from K (the addition is the usual one, while the multiplication is the composition of functions).

We fix an algebraic closure of K, denoted K^{alg}. We denote by K^{sep} the separable closure of K. We denote by $\mathbb{F}_p^{\text{alg}}$ the algebraic closure of $\mathbb{F}_p$ inside K^{alg}. We call *constants* the elements of $\mathbb{F}_p^{\text{alg}}$.

A Drinfeld module over K is a ring morphism $\phi: A \to K\{\tau\}$ such that for each $a \in A$ the coefficient of τ^0 in ϕ_a is $i(a)$, and there exists $a \in A$ such that $\phi_a \neq i(a)\tau^0$. Following the definition from [9] we call ϕ a Drinfeld module of generic characteristic if $\ker(i) = \{0\}$ and we call ϕ a Drinfeld module of finite characteristic if $\ker(i) \neq \{0\}$. In the latter case, we say that the characteristic of ϕ is $\ker(i)$ (which is a prime ideal of A). In the generic characteristic case we assume i extends to an embedding of $\text{Frac}(A)$ into K.

If $\gamma \in K^{\text{alg}} \setminus \{0\}$, we denote by $\phi^{(\gamma)}$ the Drinfeld module over K^{alg} mapping $a \in A$ to $\gamma^{-1}\phi_a\gamma$. The Drinfeld module $\phi^{(\gamma)}$ is isomorphic to ϕ over $K(\gamma)$ (see [9]).

For each field L containing K, $\phi(L)$ denotes the A-module L with the A-action given by ϕ. We call $x \in L$ a *torsion* point if there exists a nonzero $a \in A$ such that $\phi_a(x) = 0$. The set of all torsion elements of ϕ is denoted by ϕ_{tor}.

Let $\hat{h}$ be the global height associated to the Drinfeld module ϕ as in [16] (see also Section 3).

Conjecture 1.1. *Let K be a finitely generated field. For any Drinfeld module $\phi: A \to K\{\tau\}$ there exists a constant $C > 0$ such that any nontorsion point $x \in K^{\text{alg}}$ satisfies $\hat{h}(x) \geqslant \frac{C}{[K(x):K]}$.*

Before our work, the only known partial result towards Conjecture 1.1 was obtained in [3], which gave a lower bound for the canonical height of a nontorsion point $x \in K^{\text{sep}}$ restricted to the case in which ϕ is the Carlitz module.

In this paper we develop a theory of local heights $\hat{h}_v$ for Drinfeld modules over arbitrary fields of characteristic p (see Section 2). We will prove in Sections 5 and 6 certain lower bounds for the local height of an element, in case that local height is positive. Because torsion elements have all the local heights equal to 0 (see Section 2), our results represent local versions of Lehmer type inequalities for Drinfeld modules.

In all of the theorems that we will state in the present section, for a valuation v, the positive real number $d(v)$ represents the degree of the valuation v (as introduced in Section 2). For each finite extension L of K, we let M_L be the set of all discrete valuations on the field L.

In Section 5 we prove the following result.

Theorem 1.2. *Let K be a field of characteristic p and let $\phi: A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0. There exists $C > 0$ and $k \geqslant 1$, both depending only on ϕ, such that if $x \in K^{\text{alg}}$ and $v \in M_{K(x)}$, $v|v_0$ and $\hat{h}_v(x) > 0$, then*

$$\hat{h}_v(x) \geqslant \frac{C d(v)}{e(v|v_0)^{k-1}},$$

where $d(v) = \frac{d(v_0) f(v|v_0)}{[K(x):K]}$ and $e(v|v_0)$ is the ramification index and $f(v|v_0)$ is the relative degree between the residue field of v and the residue field of v_0.

Remark 1.3. With the notation from Theorem 1.2, $e(v|v_0) \leqslant [K(x):K]$ and $d(v) \geqslant \frac{d(v_0)}{[K(x):K]}$. Hence, the conclusion of Theorem 1.2 gives

$$\hat{h}_v(x) \geqslant \frac{Cd(v_0)}{[K(x):K]^k}. \tag{1}$$

Because the global height of a point is the sum of its local heights and all the local heights are nonnegative, (1) shows that if x is a nontorsion point, then

$$\hat{h}(x) \geqslant \frac{Cd(v_0)}{[K(x):K]^k}, \tag{2}$$

for some valuation v_0 of K. Moreover, the proof of Theorem 1.2 gives explicit values of C and k in terms of ϕ.

If K is a finitely generated field, then we can view K as the function field of a projective, normal variety V defined over a finite field. Then we consider the set of valuations on K associated with irreducible subvarieties of V of codimension 1. We define the degrees $d(v_0)$ in terms of intersection multiplicities (see Chapter 2 in [14]) and so, we have a positive lower bound for them depending on V. Hence, if K is finitely generated, we obtain a positive bound C_1 depending on ϕ (and V) such that if x is a nontorsion point, then

$$\hat{h}(x) \geqslant \frac{C_1}{[K(x):K]^k}. \tag{3}$$

In the case that the place v from Theorem 1.2 is not wildly ramified above K, we prove a finer result and we show this result is the best possible.

We prove in Section 6 a similar result as in Theorem 1.2 for Drinfeld modules of generic characteristic.

Theorem 1.4. *Let K be a field of characteristic p. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. There exist two positive constants C and k depending only on ϕ such that for every $x \in K^{\mathrm{alg}}$ and every place v of $K(x)$, if $\hat{h}_v(x) > 0$ and v does not lie over the place ∞ of $\operatorname{Frac}(A)$, then*

$$\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{k-1}},$$

where $v_0 \in M_K$ lies below v.

Remark 1.5. The same reasoning as in Remark 1.3 shows that if K is the function field of a variety V defined over a finite field, then there exists a positive constant C_1 depending only on ϕ (and V) such that if $x \in K^{\mathrm{alg}}$ satisfies $\hat{h}_v(x) > 0$ for some place v of $K(x)$ which does not lie over the place ∞ of $\operatorname{Frac}(A)$, then

$$\hat{h}(x) \geqslant \frac{C_1}{[K(x):K]^k}. \tag{4}$$

We show that the hypothesis from Theorem 1.4 that v does not lie over the place ∞ of $\mathrm{Frac}(A)$ cannot be removed. If v lies over ∞, then $\hat{h}_v(x)$ can be positive but arbitrarily small. Also, in case v satisfies the additional hypothesis that is not wildly ramified above K, we prove a finer inequality and show that it is best possible.

Our proof of Theorems 1.2 and 1.4 goes through a series of lemmas involving a careful analysis of the valuations at v of a point x and its iterates under the action of the Drinfeld module. In Section 4 we provide the general definitions, notation and lemmas which are valid for both finite and generic characteristic Drinfeld modules. Then in Sections 5 and 6 we treat in more depth the Lehmer conjecture for the finite and, respectively, the generic characteristic Drinfeld modules.

Extending the methods of the present paper we were able to prove in [8] (see also Chapter 7 of [6]) certain Mordell–Weil type theorems for Drinfeld modules. We were also able to prove in [8] (see also Chapter 7 of [6]) upper bounds for the size of the torsion of a Drinfeld module over certain fields. The key to our results from [6,8] is a study of the global version of the Lehmer inequality for Drinfeld modules. A brief introduction to the global height associated to a Drinfeld module is given in Section 3 of our present paper. Moreover, our height inequalities which led to the Mordell–Weil theorems of [6,8] were also used in [7] to prove a certain Mordell–Lang type statement for Drinfeld modules. Finally, because of the examples we provide in this paper (see Examples 5.13 and 6.1), we see that the results of Theorems 1.2 and 1.4 are the best that we can get towards a local version of Lehmer inequality for Drinfeld modules, which in turn shows that for proving Conjecture 1.1 one will need to employ different techniques than in the present paper, involving possibly a simultaneous analysis at all places.

2. Local heights associated to Drinfeld modules

We continue with the notation from Section 1. So, K is a field extension of $\mathbb{F}_q$ and $\phi : A \to K\{\tau\}$ is a Drinfeld module. We normalize all the discrete valuations $v \in M_K$ so that the range of v is $\mathbb{Z}$. In general, every discrete valuation we work with will have range $\mathbb{Z}$.

We associate to each valuation $v \in M_K$ a positive number $d(v)$, which we call *degree*. If L is a finite extension of K and $w \in M_L$ lies over v, i.e. $w|v$, then the degree of w is defined to be $\frac{d(v)f(w|v)}{[L:K]}$. The motivation for the degree function is given in Section 3 when we construct the global height function associated to a Drinfeld module.

Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let $v \in M_K$ and let $d(v)$ be the degree of v. For such v, we construct the local height $\hat{h}_v$ with respect to the Drinfeld module ϕ. Our construction follows [13]. We set $\tilde{v}(x) = \min\{0, v(x)\}$. For a nonconstant element $a \in A$, we define

$$V_v(x) = \lim_{n\to\infty} \frac{\tilde{v}(\phi_{a^n}(x))}{\deg(\phi_{a^n})}. \tag{5}$$

This function is well defined and satisfies the same properties as in Propositions 1–3 from [13]. Mainly, we will use the following facts:

(1) if x and all the coefficients of ϕ_a are integral at v, then $V_v(x) = 0$.
(2) for all $b \in A \setminus \{0\}$, $V_v(\phi_b(x)) = \deg(\phi_b) \cdot V_v(x)$. Moreover, we can use any nonconstant $a \in A$ for the definition of $V_v(x)$ and we will always get the same function V_v.
(3) $V_v(x \pm y) \geqslant \min\{V_v(x), V_v(y)\}$.
(4) if $x \in \phi_{\mathrm{tor}}$, then $V_v(x) = 0$.

We define then

$$\hat{h}_v(x) = -d(v)V_v(x). \tag{6}$$

Using fact (4) above, we conclude that if $x \in \phi_{\text{tor}}$, $\hat{h}_v(x) = 0$.

If L is a finite extension of K and $w \in M_L$ lies over v, then we define similarly the function V_w on L and just as above, we let $\hat{h}_w(x) = -d(w)V_w(x)$ for every $x \in L$.

3. The global height function associated to Drinfeld modules

In the present section we will show how we can put together all the local height functions corresponding to a suitable set of valuations and define the global height associated to a Drinfeld module. Our construction will put in perspective our results on the local Lehmer conjecture for Drinfeld modules, by constructing a bridge between our present paper and [8]. For more details about our construction of the global height function associated to a Drinfeld module see Chapter 4 of [6].

Definition 3.1. We call a subset $U \subset M_K$ equipped with a degree function $d: U \to \mathbb{R}_{>0}$ a *good set of valuations* if the following properties are satisfied

(i) for every nonzero $x \in K$, there are finitely many $v \in U$ such that $v(x) \neq 0$.
(ii) for every nonzero $x \in K$,

$$\sum_{v \in U} d(v) \cdot v(x) = 0.$$

When U is a good set of valuations, we will refer to property (ii) as the sum formula for U.

Definition 3.2. Let $v \in M_K$ of degree $d(v)$. We say that the valuation v is coherent (on K^{alg}) if for every finite extension L of K,

$$\sum_{\substack{w \in M_L \\ w|v}} e(w|v) f(w|v) = [L:K], \tag{7}$$

where $e(w|v)$ is the ramification index and $f(w|v)$ is the relative degree between the residue field of w and the residue field of v.

Condition (7) says that v is *defectless* in L. As before, we also let the degree of any $w \in M_L$, $w|v$ be

$$d(w) = \frac{f(w|v)d(v)}{[L:K]}. \tag{8}$$

As shown in [5, (18.1), p. 136], $v \in M_K$ is coherent if and only if for every finite extensions $L_1 \subset L_2$ of K and for every $w \in M_{L_1}$, if $w|v$, then w is defectless in L_2.

Definition 3.3. We let U_K be a good set of valuations on K. We call U_K a *coherent* good set of valuations (on K^{alg}) if for every $v \in U_K$, the valuation v is coherent (on K^{alg}).

Remark 3.4. In Definition 3.3, using the argument from p. 9 of [14], we conclude that for every finite extension L of K, if $U_L \subset M_L$ is the set of all valuations lying over valuations from U_K, then U_L is a good set of valuations.

An important example of a coherent good set of valuations is given by the set of discrete valuations associated to the irreducible divisors of a projective variety, which is regular in codimension 1. In particular, there always exist coherent good sets of valuations on finitely generated fields K. The global height $\hat{h}$ from Conjecture 1.1 is associated to such a coherent good set of valuations. For more details see [6] or [8].

If $U = U_K \subset M_K$ is a coherent good set of valuations, then for each $v \in U$, we denote by $\hat{h}_{U,v}$ the local height associated to ϕ with respect to v (the construction of $\hat{h}_{U,v}$ is the one from Section 2). Then we define the global height associated to ϕ as

$$\hat{h}_U(x) = \sum_{v \in U} \hat{h}_{U,v}(x). \tag{9}$$

For each x, the above sum is finite due to fact (1) stated above (see also Proposition 6 of [13]).

For each finite extension L of K, we let U_L be the set of all valuations of L that lie over places from U_K. Then we define the global height of x as

$$\hat{h}_{U_L}(x) = \sum_{w \in U_L} \hat{h}_{U_L,w}(x).$$

The following claim shows that our definition of the global height is independent of the field L containing x.

Claim 3.5. *Let $L_1 \subset L_2$ be finite extensions of K. Let $v \in U_{L_1}$ and $x \in L_1$. Then*

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \hat{h}_{U_{L_2},w}(x) = \hat{h}_{U_{L_1},v}(x).$$

Proof. We have

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \hat{h}_{U_{L_2},w}(x) = -\sum_{\substack{w \in U_{L_2} \\ w|v}} d(w) V_w(x).$$

Let $v_0 \in M_K$ lie below both v and w. Because $d(w) = \frac{d(v_0) f(w|v_0)}{[L_2:K]}$ and $d(v) = \frac{d(v_0) f(v|v_0)}{[L_1:K]}$ and $f(w|v_0) = f(w|v) f(v|v_0)$ and $[L_2 : K] = [L_2 : L_1] \cdot [L_1 : K]$, we conclude

$$d(w) = \frac{d(v) f(w|v)}{[L_2 : L_1]}. \tag{10}$$

Using (10) and $V_w(x) = e(w|v) V_v(x)$ we get

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \hat{h}_{U_{L_2},w}(x) = \frac{-d(v) V_v(x)}{[L_2 : L_1]} \sum_{\substack{w \in U_{L_2} \\ w|v}} e(w|v) f(w|v).$$

Because v is defectless in L_2 (because v_0 is coherent) and $\hat{h}_{U_{L_1},v}(x) = -d(v)V_v(x)$, we are done. $\square$

4. A local formulation of the Lehmer inequality for Drinfeld modules

The setting for this section is the same as before: K is a field of characteristic p, v_0 is a valuation on K of positive degree $d(v_0)$ and $\phi : A \to K\{\tau\}$ is a Drinfeld module.

The following statement would imply Conjecture 1.1 and we refer to it as the local case of the Lehmer inequality for Drinfeld modules.

Statement 4.1. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0. For the Drinfeld module $\phi : A \to K\{\tau\}$ there exists a constant $C > 0$, depending only on ϕ, such that for any $x \in K^{\mathrm{alg}}$ and any place $v|v_0$ of $K(x)$, if $\hat{h}_v(x) > 0$, then $\hat{h}_v(x) \geqslant \frac{Cd(v_0)}{[K(x):K]}$.

In Section 5 we will prove that Statement 4.1 is false but in the case of Drinfeld modules of finite characteristic we have Theorem 1.2, which we restate here.

Theorem 4.2. *Let K be a field of characteristic p and let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0. There exist $C > 0$ and $k \geqslant 1$, both depending only on ϕ, such that if $x \in K^{\mathrm{alg}}$ and $v \in M_{K(x)}$, $v|v_0$ and $\hat{h}_v(x) > 0$, then $\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{k-1}}$.*

Even though Theorem 4.2 is a local statement in which v_0 is an arbitrary valuation on K, the most interesting case is when there exists a coherent good set U of valuations on K and $v_0 \in U$. This allows us to infer global Lehmer inequalities from the local inequality provided by Theorem 4.2 as in (2). Therefore we will assume we are working in this case and point out precisely where we will use our assumption (see Remarks 5.7 and 5.10).

Before going further on, we want to point out that the field K is part of the data associated to the Drinfeld module ϕ and so, any constant C as in Theorem 4.2 might also depend on the field K. Also, at the beginning of the proof of Theorem 4.2 we will (possibly) replace K by a finite extension and we will explain how the constant C will be affected by this change. Finally, just to make things clearer, we will point out during key steps while proving Theorem 4.2 what is the dependence of C in terms of ϕ.

An immediate corollary to Theorem 4.2 is the following.

Corollary 4.3. *With the notation from Theorem* 4.2*, if L is a finite extension of $K(x)$ and $w \in M_L$ lies above v, then $\hat{h}_w(x) \geqslant \frac{Cd(w)}{e(w|v_0)^{k-1}}$.*

Proof. The proof is immediate once we note that $\hat{h}_w(x) = \frac{d(w)e(w|v)}{d(v)}\hat{h}_v(x)$ and $e(w|v_0) = e(w|v)e(v|v_0)$ and $e(w|v) \geqslant 1$. $\square$

Moreover, if p does not divide $e(v|v_0)$, then we can give a very easy expression for the exponent k in Theorem 4.2. If p does not divide $e(v|v_0)$, our value for k is optimal, as shown by Example 5.13 (see Theorem 5.15).

As Example 6.1 will show, there are infinitely many Drinfeld modules $\phi : A \to K\{\tau\}$ of generic characteristic and there exists $v_0 \in M_K$ such that for every $C > 0$ and every k, there exists $x \in K^{\mathrm{alg}}$ and there exists $v|v_0$, $v \in M_{K(x)}$ such that $0 < \hat{h}_v(x) < \frac{C}{[K(x):K]^k}$. In Theorem 6.2, we will give the best result towards Statement 4.1 for Drinfeld modules of generic characteristic.

Before proving Theorem 4.2 we will first prove a series of lemmas. The results of this section are valid for both finite and generic characteristic Drinfeld modules. As before, for each finite extension L of K and for each $v \in M_L$ such that $v|v_0$, we let $d(v) = \frac{f(v|v_0)d(v_0)}{[L:K]}$.

We first observe that if L is a finite extension of K and $v \in M_L$ lies above v_0, then

$$d(v) = \frac{d(v_0)f(v|v_0)}{[L:K]} \geqslant \frac{d(v_0)}{[L:K]}. \tag{11}$$

Hence, in proving Theorem 4.2, replacing K by a finite extension L may induce only a constant factor $[L:K]$ in the denominator of the lower bound for the local height (see Corollary 4.3 and inequality (11)).

Fix a nonconstant $t \in A$ and let $\phi_t = \sum_{i=r_0}^{r} a_i \tau^i$, where both a_{r_0} and a_r are nonzero and $0 \leqslant r_0 \leqslant r$, while $r \geqslant 1$. Theorem 4.2 is not affected if we replace ϕ by a Drinfeld module which is isomorphic to ϕ. Thus we can conjugate ϕ by an element $\gamma \in K^{\mathrm{alg}} \setminus \{0\}$ such that $\phi^{(\gamma)}$, the conjugated Drinfeld module, has the property that $\phi_t^{(\gamma)}$ is monic as a polynomial in τ. Then ϕ and $\phi^{(\gamma)}$ are isomorphic over $K(\gamma)$, which is a finite extension of K (because γ satisfies the equation $\gamma^{q^r-1}a_r = 1$).

So, we will prove Theorem 4.2 for $\phi^{(\gamma)}$ and because $\hat{h}_{\phi,v}(x) = \hat{h}_{\phi^{(\gamma)},v}(\gamma^{-1}x)$ for every place $v|v_0$ of $K(\gamma, x)$ (as proved in [13, Proposition 2]) the result will follow for ϕ.

From now on, in this section, ϕ_t is monic as a polynomial in τ.

Let L be a finite extension of K and let $v \in M_L$ be a place lying over v_0. Denote by $S = S_L$ the subset of M_L where the coefficients a_i, for $i \in \{r_0, \dots, r-1\}$, have poles. Also, denote by $S_0 = S_K$ the set of places in M_K where the coefficients a_i have poles. Thus, $v \in S$ if and only if $v_0 \in S_0$.

Definition 4.4. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module. Let L be a finite extension of K. We call $v \in M_L$ a place of *good* reduction for ϕ if for all $a \in A \setminus \{0\}$, the coefficients of ϕ_a are integral at v and the leading coefficient of ϕ_a is a unit in the valuation ring at v. If $v \in M_L$ is not a place of good reduction, we call it a place of *bad* reduction.

Lemma 4.5. *The set S_L is the set of all places in M_L at which ϕ has bad reduction.*

Proof. By the construction of the set S_L, the places in S_L are of bad reduction for ϕ. We will prove that these are all the bad places for ϕ.

Let $a \in A$. The equation $\phi_a\phi_t = \phi_t\phi_a$ will show that all the places where not all of the coefficients of ϕ_a are integral, are in S_L. Suppose this is not the case and take a place $v \notin S_L$ at which some coefficient of ϕ_a is not integral. Let $\phi_a = \sum_{i=0}^{r'} a'_i \tau^i$ and assume that i is the largest index for a coefficient a'_i that is not integral at v.

We equate the coefficient of τ^{i+r} in $\phi_a\phi_t$ and $\phi_t\phi_a$, respectively. The former is

$$a'_i + \sum_{j>i} a'_j a_{r+i-j}^{q^j} \tag{12}$$

while the latter is

$$a_i'^{q^r} + \sum_{j>i} a_{r+i-j} a_j'^{q^{r+i-j}}. \tag{13}$$

Thus the valuation at v of (12) is $v(a_i')$, because all the a_j' (for $j > i$) and a_{r+i-j} are integral at v, while $v(a_i') < 0$. Similarly, the valuation of (13) is $v(a_i'^{q^r}) = q^r v(a_i') < v(a_i')$ ($r \geqslant 1$ because t is nonconstant). This fact gives a contradiction to $\phi_a\phi_t = \phi_t\phi_a$. So, the coefficients of ϕ_a for all $a \in A$, are integral at all places of $M_L \setminus S_L$.

Now, using the same equation $\phi_a\phi_t = \phi_t\phi_a$ and equating the leading coefficients in both polynomials we obtain

$$a_{r'}' = a_{r'}'^{q^r}.$$

So, $a_{r'}' \in \mathbb{F}_p^{\mathrm{alg}}$. Thus, all the leading coefficients for polynomials ϕ_a are constants. So, if $v \in M_L \setminus S_L$, then all the coefficients of ϕ_a are integral at v and the leading coefficient of ϕ_a is a unit in the valuation ring at v for every $a \in A \setminus \{0\}$. Thus, $v \notin S_L$ is a place of good reduction for ϕ. □

Definition 4.6. For each $v \in M_L$ denote by

$$M_v = \min_{i \in \{r_0, \ldots, r-1\}} \frac{v(a_i)}{q^r - q^i} \tag{14}$$

where by convention, as always, $v(0) = +\infty$. If $r_0 = r$, definition (14) is void and in that case we define $M_v = +\infty$.

Note that $M_v < 0$ if and only if $v \in S$.

For each $v \in S$ we fix a uniformizer $\pi_v \in L$ of the place v. We define next the concept of angular component for every $y \in L \setminus \{0\}$.

Definition 4.7. Assume $v \in S$. For every nonzero $y \in L$ we define the angular component of y at v, denoted by $\mathrm{ac}_{\pi_v}(y)$, to be the residue at v of $y\pi_v^{-v(y)}$. (Note that the angular component is never 0.)

We can define in a similar manner as above the notion of angular component at each $v \in M_L$ but we will work with angular components at the places from S only.

The main property of the angular component is that for every $y, z \in L \setminus \{0\}$,

$$v(y-z) > \min\{v(y), v(z)\} = v(y) = v(z) \quad \text{if and only if}$$
$$\big(v(y), \mathrm{ac}_{\pi_v}(y)\big) = \big(v(z), \mathrm{ac}_{\pi_v}(z)\big).$$

Definition 4.8. If $v \in S$ we define P_v as the set containing $\{0\}$ and all the negatives of the integral slopes of the Newton polygon of ϕ_t, i.e. numbers of the form

$$\alpha = -\frac{v(a_i) - v(a_j)}{q^i - q^j} = \frac{v(a_i) - v(a_j)}{q^j - q^i} \in \mathbb{Z}, \tag{15}$$

for some $i \neq j$ in $\{r_0, \ldots, r\}$ such that

$$v(a_i) + q^i\alpha = v(a_j) + q^j\alpha = \min_{r_0 \leqslant l \leqslant r}\bigl(v(a_l) + q^l\alpha\bigr).$$

Definition 4.9. For each $\alpha \in P_v$ we let $l \geqslant 1$ and let $i_0 < i_1 < \cdots < i_l$ be all the indices i for which $a_i \neq 0$ and

$$v(a_i) + q^i\alpha = \min_{r_0 \leqslant j \leqslant r}\bigl(v(a_j) + q^j\alpha\bigr).$$

Then, for $j, k \in \{0, \ldots, l\}$ with $j \neq k$, we have

$$\frac{v(a_{i_j}) - v(a_{i_k})}{q^{i_k} - q^{i_j}} = \alpha. \tag{16}$$

We define $R_v(\alpha)$ as the set containing $\{1\}$ and all the nonzero solutions of the equation

$$\sum_{j=0}^{l} \mathrm{ac}_{\pi_v}(a_{i_j})X^{q^{i_j}} = 0, \tag{17}$$

where the indices i_j are the ones associated to α as in (16). Note that if $\alpha = 0$, there might be no indices i_j and i_k as in (16). In that case, the construction of $R_v(0)$ from (17) is void and so, $R_v(0) = \{1\}$. The motivation for the special case $0 \in P_v$ and $1 \in R_v(0)$ is explained in the proof of Lemma 4.16.

Proposition 4.10. *With the above definitions, $|P_v| \leqslant r - r_0 + 1$. Also, for each $\alpha \in P_v$, $|R_v(\alpha)| \leqslant q^r$.*

Proof. Clearly, $|P_v| \leqslant r - r_0 + 1$, because there are at most $(r - r_0)$ sides of the Newton polygon of ϕ_t. Clearly, for every $\alpha \in P_v$, $|R_v(\alpha)| \leqslant q^r$, because there are at most $(q^r - 1)$ nonzero solutions to (17). □

We remind the reader that our setting for this section will always be that $v_0 \in M_K$ and for a finite extension L of K, the place $v \in M_L$ lies over v_0.

Lemma 4.11. *Assume $v \in S$ and let $x \in L$. If $v(\phi_t(x)) > \min_{i \in \{r_0, \ldots, r\}} v(a_i x^{q^i})$ then $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$.*

Proof. If $v(\phi_t(x)) > \min_{i \in \{r_0, \ldots, r\}} v(a_i x^{q^i})$ it means that there exists $l \geqslant 1$ and

$$i_0 < \cdots < i_l$$

such that

$$v\bigl(a_{i_0}x^{q^{i_0}}\bigr) = \cdots = v\bigl(a_{i_l}x^{q^{i_l}}\bigr) = \min_{i \in \{r_0, \ldots, r\}} v\bigl(a_i x^{q^i}\bigr) \tag{18}$$

and also

$$\sum_{j=0}^{l} \mathrm{ac}_{\pi_v}(a_{i_j})\, \mathrm{ac}_{\pi_v}(x)^{q^{i_j}} = 0. \tag{19}$$

Equations (18) and (19) yield $v(x) \in P_v$ and $\mathrm{ac}_{\pi_v}(x) \in R_v(v(x))$, respectively, according to (15) and (17). $\square$

Lemma 4.12. *Let $v \in M_L$ and let $x \in L$. If $v(x) < \min\{0, M_v\}$, then $\hat{h}_v(x) = -d(v) \cdot v(x)$.*

Proof. For every $i \in \{r_0, \ldots, r-1\}$, $v(a_i x^{q^i}) = v(a_i) + q^i v(x) > q^r v(x)$ because $v(x) < M_v = \min_{i \in \{r_0,\ldots,r-1\}} \frac{v(a_i)}{q^r - q^i}$. This shows that $v(\phi_t(x)) = q^r v(x) < v(x) < \min\{0, M_v\}$. By induction, $v(\phi_{t^n}(x)) = q^{rn} v(x)$ for all $n \geqslant 1$. So, $V_v(x) = v(x)$ and

$$\hat{h}_v(x) = -d(v) \cdot v(x). \qquad \square$$

An immediate corollary to Lemma 4.12 is the following result.

Lemma 4.13. *Assume $v \notin S$ and let $x \in L$. If $v(x) < 0$ then $\hat{h}_v(x) = -d(v) \cdot v(x)$, while if $v(x) \geqslant 0$ then $\hat{h}_v(x) = 0$.*

Proof. First, it is clear that if $v(x) \geqslant 0$ then for all $n \geqslant 1$, $v(\phi_{t^n}(x)) \geqslant 0$ because all the coefficients of ϕ_t and thus of ϕ_{t^n} have nonnegative valuation at v. Thus $V_v(x) = 0$ and so,

$$\hat{h}_v(x) = 0.$$

Now, if $v(x) < 0$, then $v(x) < M_v$ because $M_v \geqslant 0$ ($v \notin S$). So, applying the result of Lemma 4.12 we conclude the proof of this lemma. $\square$

We will get a better insight into the local heights behavior with the following lemma.

Lemma 4.14. *Let $x \in L$. Assume $v \in S$ and $v(x) \leqslant 0$. If $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then $v(\phi_t(x)) < M_v$, unless $q = 2$, $r = 1$ and $v(x) = 0$.*

Proof. Lemma 4.11 implies that there exists $i_0 \in \{r_0, \ldots, r\}$ such that for all $i \in \{r_0, \ldots, r\}$ we have $v(a_i x^{q^i}) \geqslant v(a_{i_0} x^{q^{i_0}}) = v(\phi_t(x))$.

Suppose Lemma 4.14 is not true and so, there exists $j_0 < r$ such that

$$\frac{v(a_{j_0})}{q^r - q^{j_0}} \leqslant v\big(\phi_t(x)\big) = v(a_{i_0}) + q^{i_0} v(x).$$

This means that

$$v(a_{j_0}) \leqslant \big(q^r - q^{j_0}\big) v(a_{i_0}) + \big(q^{r+i_0} - q^{i_0+j_0}\big) v(x). \tag{20}$$

On the other hand, by our assumption about i_0, we know that $v(a_{j_0}x^{q^{j_0}}) \geqslant v(a_{i_0}x^{q^{i_0}})$ which means that

$$v(a_{j_0}) \geqslant v(a_{i_0}) + \left(q^{i_0} - q^{j_0}\right)v(x). \tag{21}$$

Putting together inequalities (20) and (21), we get

$$v(a_{i_0}) + \left(q^{i_0} - q^{j_0}\right)v(x) \leqslant \left(q^r - q^{j_0}\right)v(a_{i_0}) + \left(q^{r+i_0} - q^{i_0+j_0}\right)v(x).$$

Thus

$$v(x)\left(q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0}\right) \geqslant -v(a_{i_0})\left(q^r - q^{j_0} - 1\right). \tag{22}$$

But $q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0} = q^{r+i_0}(1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0})$ and because $j_0 < r$ and $q^{j_0-r-i_0} > 0$, we obtain

$$1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0} > 1 - q^{-1} - q^{-r} \geqslant 1 - 2q^{-1} \geqslant 0. \tag{23}$$

Also, $q^r - q^{j_0} - 1 \geqslant q^r - q^{r-1} - 1 = q^{r-1}(q-1) - 1 \geqslant 0$ with equality if and only if $q = 2$, $r = 1$ and $j_0 = 0$. We will analyze this case separately. So, as long as we are not in this special case, we do have

$$q^r - q^{j_0} - 1 > 0. \tag{24}$$

Now we have two possibilities (recall that $v(x) \leqslant 0$):

(i) $v(x) < 0$.

In this case, (22), (23) and (24) tell us that $-v(a_{i_0}) < 0$. Thus, $v(a_{i_0}) > 0$. But we know from our hypothesis on i_0 that $v(a_{i_0}x^{q^{i_0}}) \leqslant v(x^{q^r})$ which is in contradiction with the combination of the following facts: $v(x) < 0$, $i_0 \leqslant r$ and $v(a_{i_0}) > 0$.

(ii) $v(x) = 0$.

Then another use of (22), (23) and (24) gives us $-v(a_{i_0}) \leqslant 0$; thus $v(a_{i_0}) \geqslant 0$. This would mean that $v(a_{i_0}x^{q^{i_0}}) \geqslant 0$ and this contradicts our choice for i_0 because we know from the fact that $v \in S$, that there exists $i \in \{r_0, \ldots, r\}$ such that $v(a_i) < 0$. So, then we would have

$$v\left(a_i x^{q^i}\right) = v(a_i) < 0 \leqslant v\left(a_{i_0}x^{q^{i_0}}\right).$$

Thus, in either case (i) or (ii) we get a contradiction that proves the lemma except in the special case that we excluded above: $q = 2$, $r = 1$ and $j_0 = 0$. If we have $q = 2$ and $r = 1$ then

$$\phi_t(x) = a_0 x + x^2.$$

By the definition of S and because $v \in S$, $v(a_0) < 0$. Also, $M_v = v(a_0)$.

If $v(x) < 0$, then either $v(x) < M_v = v(a_0)$, in which case again $v(\phi_t(x)) < M_v$ (as shown in the proof of Lemma 4.12), or $v(x) \geqslant M_v$. In the latter case,

$$v\big(\phi_t(x)\big) = v(a_0 x) = v(a_0) + v(x) < v(a_0) = M_v.$$

So, we see that indeed, only $v(x) = 0$, $q = 2$ and $r = 1$ can make $v(\phi_t(x)) \geqslant M_v$ in the hypothesis of Lemma 4.14. □

Lemma 4.15. *Assume $v \in S$ and let $x \in L$. Excluding the case $q = 2$, $r = 1$ and $v(x) = 0$, we have that if $v(x) \leqslant 0$ then either $\hat{h}_v(x) > \frac{-d(v)M_v}{q^r}$ or $(v(x), ac_{\pi_v}(x)) \in P_v \times R_v(v(x))$.*

Proof. If $v(x) \leqslant 0$ then

$$\text{either:} \quad \text{(i)} \quad v\big(\phi_t(x)\big) < M_v,$$

in which case by Lemma 4.12 we have that $\hat{h}_v(\phi_t(x)) = -d(v) \cdot v(\phi_t(x))$. So, case (i) yields

$$\hat{h}_v(x) = -d(v) \cdot \frac{v(\phi_t(x))}{\deg \phi_t} > -d(v) \cdot \frac{M_v}{q^r} \tag{25}$$

$$\text{or:} \quad \text{(ii)} \quad v\big(\phi_t(x)\big) \geqslant M_v,$$

in which case, Lemma 4.14 yields

$$v\big(\phi_t(x)\big) > v\big(a_{i_0} x^{q^{i_0}}\big) = \min_{i \in \{r_0, \dots, r\}} v\big(a_i x^{q^i}\big). \tag{26}$$

Using (26) and Lemma 4.11 we conclude that case (ii) yields $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$. □

Now we analyze the excluded case from Lemma 4.15.

Lemma 4.16. *Assume $v \in S$ and let $x \in L$. If $v(x) \leqslant 0$, then either*

$$\big(v(x), ac_{\pi_v}(x)\big) \in P_v \times R_v\big(v(x)\big)$$

or $\hat{h}_v(x) \geqslant \frac{-d(v)M_v}{q^r}$.

Proof. Using the result of Lemma 4.15 we have left to analyze the case: $q = 2$, $r = 1$ and $v(x) = 0$.

As shown in the proof of Lemma 4.14, in this case $\phi_t(x) = a_0 x + x^2$ and

$$v\big(\phi_t(x)\big) = v(a_0) = M_v < 0.$$

Then, either $v(\phi_{t^2}(x)) = v(\phi_t(x)^2) = 2M_v < M_v$ or $v(\phi_{t^2}(x)) > v(a_0 \phi_t(x)) = v(\phi_t(x)^2)$. If the former case holds, then by Lemma 4.12,

$$\hat{h}_v\big(\phi_{t^2}(x)\big) = -d(v) \cdot 2M_v$$

and so,

$$\hat{h}_v(x) = \frac{-d(v) \cdot 2M_v}{4}.$$

If the latter case holds, i.e. $v(\phi_t(\phi_t(x))) > v(a_0\phi_t(x)) = v(\phi_t(x)^2)$, then $\mathrm{ac}_{\pi_v}(\phi_t(x))$ satisfies the equation

$$\mathrm{ac}_{\pi_v}(a_0)X + X^2 = 0.$$

Because the angular component is never 0, it must be that $\mathrm{ac}_{\pi_v}(\phi_t(x)) = \mathrm{ac}_{\pi_v}(a_0)$ (recall that we are working now in characteristic 2). But, because $v(a_0x) < v(x^2)$ we can relate the angular component of x and the angular component of $\phi_t(x)$ and so,

$$\mathrm{ac}_{\pi_v}(a_0) = \mathrm{ac}_{\pi_v}\big(\phi_t(x)\big) = \mathrm{ac}_{\pi_v}(a_0x) = \mathrm{ac}_{\pi_v}(a_0)\,\mathrm{ac}_{\pi_v}(x).$$

This means $\mathrm{ac}_{\pi_v}(x) = 1$ and so, the excluded case amounts to a dichotomy similar to the one from Lemma 4.15: either $(v(x), \mathrm{ac}_{\pi_v}(x)) = (0, 1)$ or $\hat{h}_v(x) = \frac{-d(v)M_v}{2}$. The definitions of P_v and $R_v(\alpha)$ from (15) and (17), respectively, yield that $(0, 1) \in P_v \times R_v(0)$. □

Finally, we note that in Lemma 4.16 we have

$$-\frac{d(v)M_v}{q^r} = -\frac{d(v)e(v|v_0)M_{v_0}}{q^r}.$$

We have obtained the following dichotomy (see also Proposition 4.10).

Lemma 4.17. *Assume $v \in S$ and let $x \in L$. If $v(x) \leqslant 0$ then either*

$$\hat{h}_v(x) \geqslant \frac{-d(v)e(v|v_0)M_{v_0}}{q^r}$$

or

$$\big(v(x), \mathrm{ac}_{\pi_v}(x)\big) \in P_v \times R_v\big(v(x)\big)$$

with $|P_v| \leqslant r - r_0 + 1$ and for each $\alpha \in P_v$, $|R_v(\alpha)| \leqslant q^r$.

The following lemma shows that if $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$, then $v(\phi_t(x))$ is determined completely only in terms of $v(x)$.

Lemma 4.18. *There are no x and x' in L verifying the following properties*

(a) $v(x) \neq v(x')$;
(b) $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ *and* $(v(x'), \mathrm{ac}_{\pi_v}(x')) \notin P_v \times R_v(v(x'))$;
(c) $v(\phi_t(x)) = v(\phi_t(x'))$.

Proof. Condition (b) and Lemma 4.11 yield

$$v\bigl(\phi_t(x)\bigr) = \min_{r_0 \leqslant i \leqslant r} v\bigl(a_i x^{q^i}\bigr)$$

and

$$v\bigl(\phi_t(x')\bigr) = \min_{r_0 \leqslant i \leqslant r} v\bigl(a_i x'^{q^i}\bigr).$$

Then the conclusion of our lemma is immediate because the function

$$F(y) = \min_{r_0 \leqslant i \leqslant r} v\bigl(a_i y^{q^i}\bigr)$$

is a strictly increasing piecewise linear function. □

Lemma 4.19. *Assume $v \in S$. Given (α_1, γ_1), there are at most q^r possible values of $\mathrm{ac}_{\pi_v}(x)$ when x ranges over nonzero elements of L such that $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ and $(\alpha_1, \gamma_1) = (v(\phi_t(x)), \mathrm{ac}_{\pi_v}(\phi_t(x)))$.*

Proof. Indeed, we saw in Lemma 4.18 that $v(x)$ is uniquely determined given $\alpha_1 = v(\phi_t(x))$ under the hypothesis of Lemma 4.19. We also have

$$\mathrm{ac}_{\pi_v}\bigl(\phi_t(x)\bigr) = \sum_j \mathrm{ac}_{\pi_v}(a_{i_j})\, \mathrm{ac}_{\pi_v}(x)^{q^{i_j}}, \tag{27}$$

where i_j runs through a prescribed subset of $\{r_0, \dots, r\}$ corresponding to those i such that $v(a_i) + q^i v(x) = v(\phi_t(x)) = \min_{i \in \{r_0,\dots,r\}} v(a_i x^{q^i})$. This subset of indices i_j depends only on $v(x)$, which in turn, depends only on $\alpha_1 = v(\phi_t(x))$. So, there are at most q^r possible values for $\mathrm{ac}_{\pi_v}(x)$ to solve (27) given $\gamma_1 = \mathrm{ac}_{\pi_v}(\phi_t(x))$. □

5. The finite characteristic case

We continue with the notation from the previous section. Hence, $\phi : A \to K\{\tau\}$ is a Drinfeld module. Also, for some $t \in A$,

$$\phi_t = \sum_{i=r_0}^{r} a_i \tau^i,$$

with $a_r = 1$. As before we let L be a finite extension of K and let $x \in L$. Finally, S, M_v, P_v, R_v are defined as in Section 4. In this section, unless otherwise stated, we will assume that

$$\boxed{r_0 \geqslant 1\text{, i.e. } \phi \text{ has finite characteristic and } \phi_t \text{ is inseparable.}}$$

Because for every Drinfeld module of finite characteristic we can find a nonconstant $t \in A$ such that ϕ_t is inseparable, the above boxed condition will always be achieved for some $t \in A$, in the case of Drinfeld modules of finite characteristic.

Our strategy for proving Theorem 4.2 will be to prove that if $\hat{h}_v(x) > 0$ then *either*

$$\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{\frac{r}{r_0}-1}},$$

where $C > 0$ is a constant depending only on ϕ, *or*

$$v \in S \quad \text{and} \quad \big(v(x), \mathrm{ac}_{\pi_v}(x)\big) \text{ belongs to a set of cardinality we can control.}$$

Lemma 5.1. *If $v \in S$ define $N_v := \max\{\frac{-v(a_i)}{q^i-1} \mid r_0 \leqslant i \leqslant r\}$ (recall our convention $v(0) = +\infty$). If $v(x) \geqslant N_v$, then $\hat{h}_v(x) = 0$.*

Proof. Using the definition of N_v, if $v(x) \geqslant N_v$ then $q^i v(x) + v(a_i) \geqslant v(x)$, for every i. Hence

$$v\big(\phi_t(x)\big) \geqslant \min_{1 \leqslant i \leqslant r} \big\{q^i v(x) + v(a_i)\big\} \geqslant v(x) \geqslant N_v.$$

By induction, we get that $v(\phi_{t^n}(x)) \geqslant N_v$ for all $n \geqslant 1$, which yields that $V_v(x) = 0$ and so,

$$\hat{h}_v(x) = 0. \qquad \square$$

Thus, if $v \in S$ and $\hat{h}_v(x) > 0$ it must be that $v(x) < N_v$.

Lemma 5.2. *Assume $v \in S$ and let $x \in L$. If $v(x) < N_v$ and if $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then $v(\phi_t(x)) < v(x)$. Moreover, if $i_0 \in \{r_0, \ldots, r\}$ such that $v(a_{i_0} x^{q^{i_0}}) = v(\phi_t(x))$, then $v(x) < \frac{-v(a_{i_0})}{q^{i_0}-1}$.*

Proof. Indeed, by the hypothesis and by Lemma 4.11, there exists $i_0 \in \{r_0, \ldots, r\}$ such that for all $i \in \{r_0, \ldots, r\}$,

$$v(a_{i_0}) + q^{i_0} v(x) = v\big(\phi_t(x)\big) \leqslant v(a_i) + q^i v(x). \tag{28}$$

If $v(\phi_t(x)) \geqslant v(x)$ then, using (28), we get that

$$v(x) \leqslant v(a_i) + q^i v(x)$$

which implies that $v(x) \geqslant -\frac{v(a_i)}{q^i-1}$ for every i. Thus

$$v(x) \geqslant N_v,$$

contradicting the hypothesis of our lemma. So, we must have $v(\phi_t(x)) < v(x)$. In particular, we also get that $v(a_{i_0}) + q^{i_0} v(x) < v(x)$, i.e.

$$v(x) < \frac{-v(a_{i_0})}{q^{i_0} - 1}. \qquad \square \tag{29}$$

Our goal is to establish a dichotomy similar to the one from Lemma 4.17 under the following hypothesis:

$$\boxed{v \in S,\ x \in L,\ \hat{h}_v(x) > 0 \text{ and } 0 < v(x) < N_v.}$$

In Lemma 5.2 we saw that if $v(x) < N_v$ then either $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ or $v(\phi_t(x)) < v(x)$. In the latter case, if $v(\phi_t(x)) > 0$ we apply then the same reasoning to $\phi_t(x)$ and derive that either $(v(\phi_t(x)), \mathrm{ac}_{\pi_v}(\phi_t(x))) \in P_v \times R_v(v(\phi_t(x)))$ or $v(\phi_{t^2}(x)) < v(\phi_t(x))$. We repeat this analysis and, since $v(x)$ and all $v(a_i)$ are integers, after a finite number of steps, say n, we must have that either

$$v\big(\phi_{t^n}(x)\big) \leqslant 0$$

or

$$\big(v\big(\phi_{t^n}(x)\big), \mathrm{ac}_{\pi_v}\big(\phi_{t^n}(x)\big)\big) \in P_v \times R_v\big(v\big(\phi_{t^n}(x)\big)\big).$$

But we analyzed in Lemma 4.17 what happens to the cases in which, for an element y of positive local height at v, $v(y) \leqslant 0$. We obtained that either

$$\hat{h}_v(y) \geqslant \frac{-d(v)M_{v_0}e(v|v_0)}{q^r} \tag{30}$$

or

$$\big(v(y), \mathrm{ac}_{\pi_v}(y)\big) \in P_v \times R_v\big(v(y)\big) \tag{31}$$

and $|P_v| \leqslant r - r_0 + 1 \leqslant r$ because $r_0 \geqslant 1$.

We will use repeatedly Eqs. (30) and (31) for $y = \phi_{t^n}(x)$. So, if (30) holds for $y = \phi_{t^n}(x)$ then

$$\hat{h}_v(x) \geqslant \frac{-d(v)M_{v_0}e(v|v_0)}{q^{rn}q^r}. \tag{32}$$

We will see next what happens if (31) holds. We can go back through the steps that we made in order to get to (31) and see that actually $v(x)$ and $\mathrm{ac}_{\pi_v}(x)$ belong to prescribed sets of cardinality independent of n.

Lemma 5.3. *Assume $v \in S$ and suppose that $v(x) < N_v$. If*

$$\big(v\big(\phi_{t^k}(x)\big), \mathrm{ac}_{\pi_v}\big(\phi_{t^k}(x)\big)\big) \notin P_v \times R_v\big(v\big(\phi_{t^k}(x)\big)\big)$$

for $0 \leqslant k \leqslant n-1$, then for each value

$$(\alpha_n, \gamma_n) = \big(v\big(\phi_{t^n}(x)\big), \mathrm{ac}_{\pi_v}\big(\phi_{t^n}(x)\big)\big),$$

the valuation of x is uniquely determined and $\mathrm{ac}_{\pi_v}(x)$ belongs to a set of cardinality at most q^{r^2-r}.

Proof. The fact that $v(x)$ is uniquely determined follows after n successive applications of Lemma 4.18 to $\phi_{t^{n-1}}(x), \ldots, \phi_t(x), x$.

Because $(v(\phi_{t^k}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))) \notin P_v \times R_v(v(\phi_{t^k}(x)))$ for $k < n$, then Lemma 4.11 shows that for each $k < n$ we are solving an equation of the form

$$\sum_j \mathrm{ac}_{\pi_v}(a_{i_j})\, \mathrm{ac}_{\pi_v}\big(\phi_{t^k}(x)\big)^{q^{i_j}} = \mathrm{ac}_{\pi_v}\big(\phi_{t^{k+1}}(x)\big) \tag{33}$$

in order to express $\mathrm{ac}_{\pi_v}(\phi_{t^k}(x))$ in terms of $\mathrm{ac}_{\pi_v}(\phi_{t^{k+1}}(x))$ for each $k < n$. Equations (33) are uniquely determined by the sets of indices $i_j \in \{r_0, \ldots, r\}$ which in turn are uniquely determined by $v(\phi_{t^k}(x))$, i.e. for each k and each corresponding index i_j

$$v\big(a_{i_j}\phi_{t^k}(x)^{q^{i_j}}\big) = \min_{i \in \{r_0, \ldots, r\}} v\big(a_i \phi_{t^k}(x)^{q^i}\big). \tag{34}$$

Using the result of Lemma 5.2 and the hypothesis of our lemma, we see that

$$v(x) > v\big(\phi_t(x)\big) > v\big(\phi_{t^2}(x)\big) > \cdots > v\big(\phi_{t^n}(x)\big) \tag{35}$$

and so the equations from (33) appear in a prescribed order. Now, in most of the cases, these equations will consist of only one term on their left-hand side; i.e. they will look like

$$\mathrm{ac}_{\pi_v}(a_{i_0})\, \mathrm{ac}_{\pi_v}\big(\phi_{t^k}(x)\big)^{q^{i_0}} = \mathrm{ac}_{\pi_v}\big(\phi_{t^{k+1}}(x)\big). \tag{36}$$

Equation (36) has a unique solution. The other equations of type (33) but not of type (36) are associated to some of the values of $v(\phi_{t^k}(x)) \in P_v$. Indeed, according to the definition of P_v from (15), only for those values (of the slopes of the Newton polygon of ϕ_t) we can have for $i \neq i'$

$$v(a_i) + q^i v(x) = v(a_{i'}) + q^{i'} v(x) \tag{37}$$

and so, both indices i and i' can appear in (33).

Thus the number of equations of type (33) but not of type (36) is at most $r - 1$, because there are at most $r - r_0$ different segments (with different slopes) in the Newton polygon of ϕ_t (and also, remember that we are working under the assumption that ϕ_t is inseparable, i.e. $r_0 \geqslant 1$). Moreover, these equations will appear in a prescribed order, each not more than once, because of (35). These observations determine the construction of the finite set that will contain all the possible values for $\mathrm{ac}_{\pi_v}(x)$, given $\gamma_n = \mathrm{ac}_{\pi_v}(\phi_{t^n}(x))$. An equation of type (33) can have at most q^r solutions; thus $\mathrm{ac}_{\pi_v}(x)$ lives in a set of cardinality at most q^{r^2-r}. $\square$

Because of the result of Lemma 5.3, we know that we can construct in an unique way $v(x)$ given $v(\phi_{t^n}(x))$ and the fact that for every $j < n$, $\phi_{t^j}(x)$ does not satisfy (31).

Definition 5.4. With the notation as in Lemma 5.3, for each n there are at most $|P_v|$ values for $v(x)$ such that

$$\big(v\big(\phi_{t^n}(x)\big), \mathrm{ac}_{\pi_v}\big(\phi_{t^n}(x)\big)\big) \in P_v \times R_v\big(v\big(\phi_{t^n}(x)\big)\big) \tag{38}$$

and (38) does not hold for $n' < n$. We denote by $P_v(n)$ this set of values for $v(x)$. Clearly $P_v(0) = P_v$.

Lemma 5.3 yields that for each fixed $(\alpha_n, \gamma_n) \in P_v \times R_v(\alpha_n)$, there are at most q^{r^2-r} possible values for $\mathrm{ac}_{\pi_v}(x)$ such that

$$\bigl(v\bigl(\phi_{t^n}(x)\bigr), \mathrm{ac}_{\pi_v}\bigl(\phi_{t^n}(x)\bigr)\bigr) = (\alpha_n, \gamma_n)$$

and $\phi_{t^j}(x)$ does not satisfy (31) for $j < n$.

Definition 5.5. With the above notation, for $\alpha = v(x) \in P_v(n)$ we define by $R_{v,n}(\alpha)$ the set of all possible values for $\mathrm{ac}_{\pi_v}(x)$ such that n is the smallest integer for which (38) holds. Clearly, $R_{v,0} = R_v$.

Let $\alpha \in P_v(n)$ and so, $v(\phi_{t^n}(x)) = \alpha_n \in P_v$. Using the definition of $R_v(\alpha_n)$ for $\alpha_n \in P_v$ from (17), we get

$$\bigl|R_{v,0}\bigl(v\bigl(\phi_{t^n}(x)\bigr)\bigr)\bigr| \leqslant q^r. \tag{39}$$

Inequality (39) and the result of Lemma 5.3 gives the estimate:

$$\bigl|R_{v,n}(\alpha)\bigr| \leqslant \bigl|R_{v,0}\bigl(v\bigl(\phi_{t^n}(x)\bigr)\bigr)\bigr| \cdot q^{r^2-r} \leqslant q^r \cdot q^{r^2-r} = q^{r^2} \tag{40}$$

for every $\alpha \in P_v(n)$ and for every $n \geqslant 0$.

Now, we estimate the magnitude of n, i.e. the number of steps that we need to make starting with $0 < v(x) < N_v$ such that in the end $\phi_{t^n}(x)$ satisfies either (30) or (31).

Lemma 5.6. *Assume $v \in S$ and $\hat{h}_v(x) > 0$. Then there exists a positive constant c_1 depending only on ϕ and v_0 (the place lying below v), and there exists an integer m bounded above in terms of ϕ and $e(v|v_0)$ such that either for some $n \leqslant m$, $v(x) \in P_v(n)$ and $\mathrm{ac}_{\pi_v}(x) \in R_{v,n}(v(x))$, or $\hat{h}_v(x) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$.*

Proof. If (31) does not hold for x then we know that there exists $i_0 \geqslant r_0$ such that $v(\phi_t(x)) = q^{i_0} v(x) + v(a_{i_0})$.

Now, if $\phi_t(x)$ also does not satisfy (31) then for some i_1

$$v\bigl(\phi_{t^2}(x)\bigr) = q^{i_1} v\bigl(\phi_t(x)\bigr) + v(a_{i_1}) \leqslant q^i v\bigl(\phi_t(x)\bigr) + v(a_i)$$

for all $i \in \{r_0, \ldots, r\}$. So, in particular

$$v\bigl(\phi_{t^2}(x)\bigr) \leqslant q^{i_0} v\bigl(\phi_t(x)\bigr) + v(a_{i_0}) \tag{41}$$

and in general

$$v\bigl(\phi_{t^{k+1}}(x)\bigr) \leqslant q^{i_0} v\bigl(\phi_{t^k}(x)\bigr) + v(a_{i_0}) \tag{42}$$

if $(v(\phi_{t^k}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))) \notin P_v \times R_v(v(\phi_{t^k}(x)))$. Let us define the sequence $(y_j)_{j\geqslant 0}$ by

$$y_0 = v(x) \quad \text{and} \quad \text{for all } j \geqslant 1: \quad y_j = q^{i_0} y_{j-1} + v(a_{i_0}).$$

If $\phi_{t^i}(x)$ does not satisfy (31) for $i \in \{0, \ldots, n-1\}$ then by (42),

$$y_n \geqslant v\big(\phi_{t^n}(x)\big). \tag{43}$$

The sequence $(y_j)_{j\geqslant 0}$ can be easily computed and we see that

$$y_j = q^{i_0 j}\left(v(x) + \frac{v(a_{i_0})}{q^{i_0}-1}\right) - \frac{v(a_{i_0})}{q^{i_0}-1}. \tag{44}$$

But $v(x) < -\frac{v(a_{i_0})}{q^{i_0}-1}$, as a consequence of $v(x) < N_v$ and Lemma 5.2 (see Eq. (29)). Thus,

$$v(x) + \frac{v(a_{i_0})}{q^{i_0}-1} \leqslant -\frac{1}{q^{i_0}-1} \tag{45}$$

because $v(x), v(a_{i_0}) \in \mathbb{Z}$. Using inequality (45) in the formula (44) we get

$$y_j \leqslant \frac{1}{q^{i_0}-1}\big(-q^{i_0 j} - v(a_{i_0})\big). \tag{46}$$

We define

$$c_{v_0} = \max\big\{-v_0(a_i) \mid r_0 \leqslant i \leqslant r\big\}. \tag{47}$$

So, $c_{v_0} \geqslant 1$ because we know that at least one of the a_i has a pole at v, thus at v_0 (we are working under the assumption that $v \in S$). Clearly, c_{v_0} depends only on v_0 and ϕ (the dependence on K is part of the Drinfeld module data for ϕ). For simplicity, we denote c_{v_0} by c. Because of the definition of c, we have

$$-v(a_{i_0}) \leqslant e(v|v_0)c. \tag{48}$$

Now, if we pick m minimal such that

$$q^{r_0 m} \geqslant ce(v|v_0) \tag{49}$$

then we see that m depends only on ϕ and $e(v|v_0)$. Using that $i_0 \geqslant r_0$ we get that

$$q^{i_0 m} \geqslant ce(v|v_0). \tag{50}$$

So, using inequalities (46), (48) and (50) we obtain $y_m \leqslant 0$. Because of (43) we derive that

$$v\big(\phi_{t^m}(x)\big) \leqslant 0$$

which according to the dichotomy from Lemma 4.17 yields that $\phi_{t^m}(x)$ satisfies either (30) or (31). Thus, we need at most m steps to get from x to some $\phi_{t^n}(x)$ for which one of the two equations (30) or (31) is valid. This means that either

$$\hat{h}_v(x) \geqslant \frac{-d(v)M_{v_0}e(v|v_0)}{q^{rm}q^r} \quad \bigl(\text{which holds if (30) is valid after } n \leqslant m \text{ steps}\bigr), \tag{51}$$

or

$$\phi_{t^n}(x) \quad \text{satisfies (31) for } n \leqslant m. \tag{52}$$

This last equation implies that $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v(n) \times R_{v,n}(v(x))$ for some $n \leqslant m$.

We analyze now the inequality from Eq. (51). By the minimality of m satisfying (49), we have

$$q^{rm} = \bigl(q^{r_0(m-1)}\bigr)^{\frac{r}{r_0}} q^r < \bigl(ce(v|v_0)\bigr)^{\frac{r}{r_0}} q^r. \tag{53}$$

So, if (51) holds, we have the following inequality

$$\hat{h}_v(x) > \frac{-d(v)M_{v_0}e(v|v_0)}{c^{\frac{r}{r_0}} q^{2r} e(v|v_0)^{\frac{r}{r_0}}}. \tag{54}$$

To simplify the notation in the future we introduce new constants c_i, that will always depend only on v_0 and ϕ (and K). For example, M_{v_0} is a negative number which is at most $-\frac{1}{q^r-1}$ and so, (54) says that

$$\hat{h}_v(x) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}} \quad \text{or for some } n \leqslant m, \bigl(v(x), \mathrm{ac}_{\pi_v}(x)\bigr) \in P_v(n) \times R_{v,n}\bigl(v(x)\bigr). \tag{55}$$

Moreover, $|R_{v,n}(v(x))| \leqslant q^{r^2}$. $\square$

Remark 5.7. If K is the function field of a projective, normal variety V defined over a finite field, and we construct the set of valuations associated to irreducible subvarieties of codimension 1 in V, then there are finitely many places of bad reduction in K and so, we can choose $c_1 > 0$ as in Lemma 5.6 bounded from below only in terms of ϕ (by taking the minimum over all the finitely many positive constants we obtain for the places of bad reduction in K).

For the convenience of the reader we restate the exact findings of Lemma 5.6 in a separate corollary (see also the discussion from Remark 5.7).

Corollary 5.8. *Assume $v \in S$ and $\hat{h}_v(x) > 0$. Let $c = \max_i\{-v_0(a_i)\}$. Let m be the least integer such that $q^{r_0 m} \geqslant ce(v|v_0)$. There exists a positive constant c_1 depending only on ϕ such that either for some $n \leqslant m$,*

$$\bigl(v(x), \mathrm{ac}_{\pi_v}(x)\bigr) \in P_v(n) \times R_{v,n}\bigl(v(x)\bigr)$$

or $\hat{h}_v(x) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$. Moreover, if the former case holds, then $|R_{v,n}(v(x))| \leqslant q^{r^2}$.

The following combinatorial lemma will be used in the proof of Theorem 4.2.

Lemma 5.9. *Let L be a field extension of $\mathbb{F}_q$ and let v be a discrete valuation on L. Let I be the sequence of integers consisting of*

$$\alpha_1^{(1)} = \alpha_1^{(2)} = \cdots = \alpha_1^{(l_1)} < \alpha_2^{(1)} = \cdots = \alpha_2^{(l_2)} < \cdots < \alpha_g^{(1)} = \cdots = \alpha_g^{(l_g)},$$

where each $l_1, \ldots, l_g$ are positive integers. Let $l := \sum_{j=1}^{g} l_j$ be the cardinality of the sequence I.

Let N be an integer greater than or equal to all the elements of I. For each $\alpha \in I$, let $R(\alpha)$ be a nonempty finite set of nonzero elements of the residue field at v. Let W be an $\mathbb{F}_q$-vector subspace of L with the property that for all $w \in W$, if $v(w) \leqslant N$, then there exist $\alpha_j^{(i)} \in I$, such that $v(w) = \alpha_j^{(i)}$ and $\mathrm{ac}_{\pi_v}(w) \in R(\alpha_j^{(i)})$.

Let f be the smallest positive integer greater than or equal to $\max_{\alpha \in I} \log_q |R(\alpha)|$. Then the $\mathbb{F}_q$-codimension of $\{w \in W \mid v(w) > N\}$ is bounded above by $l \cdot f$.

Proof. Let $\alpha_{(g+1)}^{(1)} = N + 1$. For $1 \leqslant j \leqslant g + 1$, define $W_j = \{w \in W \mid v(w) \geqslant \alpha_j^{(1)}\}$. For $1 \leqslant j \leqslant g$, the hypothesis gives an injection

$$W_j / W_{j+1} \to \bigcup_{i=1}^{l_j} R\big(\alpha_j^{(i)}\big) \cup \{0\}$$

taking w to the residue of $w/\pi_v^{\alpha_j^{(1)}}$. Thus

$$q^{\dim_{\mathbb{F}_q} W_j / W_{j+1}} \leqslant l_j q^f + 1 < q^{l_j f + 1},$$

so $\dim_{\mathbb{F}_q} W_j / W_{j+1} \leqslant l_j f$ (note that we used the fact that $f > 0$ in order to have the inequality $l_j q^f + 1 < q^{l_j f + 1}$). Summing over j gives $\dim_{\mathbb{F}_q} W_1 / W_{g+1} \leqslant lf$, as desired. □

We are ready to prove Theorem 4.2.

Proof of Theorem 4.2. From our assumption, $\hat{h}_v(x) > 0$. Because ϕ is a Drinfeld module of finite characteristic, there exists a nonconstant $t \in A$ such that ϕ_t is inseparable.

First we observe that if $v \notin S$ then by Lemma 4.13, $v(x) < 0$ (otherwise $\hat{h}_v(x) = 0$). Then, using again Lemma 4.13, we get $\hat{h}_v(x) = -d(v) \cdot v(x) \geqslant d(v)$, as $v(x) < 0$. Hence, if $v \notin S$, we already obtained the desired inequality (with $k = 1$ and $C = 1$). So, from now on we suppose that the valuation v is in S.

Let f be the smallest positive integer such that

$$f \geqslant \max_{\substack{n \leqslant m \\ \alpha \in P_v(n)}} \log_q \big|R_{v,n}(\alpha)\big|.$$

So $f \leqslant r^2$, as shown by Corollary 5.8. We also have the following inequality

$$z := \sum_{n=0}^{m} \big|P_v(n)\big| \leqslant r(m+1) \quad \big(\text{because } \big|P_v(n)\big| \leqslant r \text{ for every } n\big). \tag{56}$$

Let $W = \mathrm{Span}(\{x, \phi_t(x), \ldots, \phi_{t^{zf}}(x)\})$. Because $\hat{h}_v(x) > 0$ we know that $x \notin \phi_{\mathrm{tor}}$ and so, $\dim_{\mathbb{F}_q} W = 1 + zf$. We also get from $\hat{h}_v(x) > 0$ that for all $0 \neq w \in W$, $\hat{h}_v(w) > 0$. Then by Lemma 5.1, we get that for all $0 \neq w \in W$, $v(w) \leqslant N_v - 1$.

We apply Lemma 5.9 to W with I being the sequence of all elements in $\bigcup_{0 \leqslant n \leqslant m} P_v(n)$ (appearing as many times as they appear in the different sets $P_v(n)$), $R(\alpha) = R_{v,n}(\alpha)$ (for each α in each set $P_v(n)$) and $N = N_v - 1$. Because $z = \sum_{n=0}^{m} |P_v(n)|$ and $f \geqslant \max_{\alpha \in I} \log_q |R(\alpha)|$, we conclude that there exists $0 \neq b \in \mathbb{F}_q[t]$, of degree at most zf in t such that

$$\big(v\big(\phi_b(x)\big), \mathrm{ac}_{\pi_v}\big(\phi_b(x)\big)\big) \notin \bigcup_{0 \leqslant n \leqslant m} \big(P_v(n) \times R_{v,n}\big(v\big(\phi_b(x)\big)\big)\big). \tag{57}$$

We know that $\hat{h}_v(x) > 0$ and so $\hat{h}_v(\phi_b(x)) > 0$. Equations (57) and (55) yield

$$\hat{h}_v\big(\phi_b(x)\big) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0} - 1}}.$$

Thus

$$\hat{h}_v(x) > \frac{c_1 d(v)}{q^{r \deg(b)} e(v|v_0)^{\frac{r}{r_0} - 1}}.$$

But, using inequality (56), we obtain

$$q^{r \deg(b)} \leqslant q^{rzf} \leqslant q^{r^4(m+1)} = q^{r^4} \big(q^{rm}\big)^{r^3}.$$

We use (53) and we get

$$q^{r \deg(b)} < q^{r^4} \big(ce(v|v_0)\big)^{\frac{r}{r_0} \cdot r^3} q^{r^4}.$$

Thus there exists a constant $C > 0$ depending only on c_1, c, q and r such that

$$\hat{h}_v(x) > \frac{C d(v)}{e(v|v_0)^{\frac{r^4 + r}{r_0} - 1}}. \tag{58}$$

Because c_1 and c depend only on ϕ we get the conclusion of Theorem 4.2. □

Remark 5.10. From the above proof we see that the constant C depends only on q, r and the numbers $v_0(a_i)$ for $r_0 \leqslant i \leqslant r - 1$, under the hypothesis that ϕ_t is monic as a polynomial in τ. As we said before, for the general case, when ϕ_t is not necessarily monic, the constant C from Theorem 4.2 will be multiplied by the inverse of the degree of the extension of K that we have to allow in order to construct an isomorphic Drinfeld module $\phi^{(\gamma)}$ for which $\phi_t^{(\gamma)}$ is monic. The degree of this extension is at most $(q^r - 1)$ because $\gamma^{q^r - 1} a_r = 1$.

Finally, we note that the dependence on the numbers $v_0(a_i)$ could be considered simply a dependence on ϕ. This dependence appears only if v_0 is a place of bad reduction for ϕ and so, for each coherent good set of valuations on K there are finitely many such places of bad reduction (see also Remark 5.7).

Remark 5.11. It is interesting to note that (58) shows that Statement 4.1 holds, i.e. $k = 1$, in the case that $e(v|v_0) = 1$, which is the case when x belongs to an unramified extension above v_0. Also, as observed in the beginning of the proof of Theorem 4.2, if v and so, equivalently v_0 is not a pole for any of the a_i then we automatically get exponent $k = 1$ in Theorem 4.2, as proved in Lemma 4.13.

So, we see that in the course of proving Theorem 4.2 we got an even stronger result that allows us to conclude that Statement 4.1 and so, implicitly Conjecture 1.1 hold in the maximal extension unramified above the places of bad reduction for ϕ.

Remark 5.12. Also, it is interesting to note that the above proof of Theorem 4.2 shows that for every place v associated to L (as in Section 2), there exists a number n depending only on r and $e(v|v_0)$ so that there exists $b \in \mathbb{F}_q[t]$ of degree at most n in t for which either $v(\phi_b(x)) < M_v$ (in which case $\hat{h}_v(x) > 0$), or $v(\phi_b(x)) \geqslant N_v$ (in which case $\hat{h}_v(x) = 0$).

Example 5.13. The result of Theorem 4.2 is optimal in the sense that we cannot hope to get the conjectured Lehmer inequality for the local height, i.e. $\frac{C}{d}$ (where d is the degree of the extension). We can only get, in the general case for the local height, an inequality with some exponent $k > 1$, i.e. $\frac{C}{d^k}$.

For example, take $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$ and define for some $r \geqslant 2$,

$$\phi_t = \tau^r - t^{1-q}\tau.$$

Let $d = q^m - 1$, for some $m \geqslant r$. Then let $x = t\alpha$ where α is a root of

$$\alpha^d - \alpha - \frac{1}{t} = 0.$$

Then $L = K(x)$ is totally ramified above t of degree d. Let v be the unique valuation of L for which $v(t) = d$. We compute

$$\begin{aligned}
&P_v = \{0\} \quad \text{because the slope } \frac{-d(q-1)}{q^r - q} \notin \mathbb{Z} \text{ as } (q, d) = 1,\\
&M_v = -\frac{d(q-1)}{q^r - q},\\
&N_v = d,\\
&v(x) = d - 1 = q^m - 2.
\end{aligned}$$

We compute easily $v(\phi_{t^i}(x)) = d - q^i$ for every $i \in \{0, \dots, m\}$. Furthermore, $v(\phi_{t^m}(x)) = d - q^m = -1 \notin P_v$. Thus $v(\phi_{t^m}(x))$ is negative and not in P_v and so, Lemma 4.14 yields

$$v\big(\phi_{t^{m+1}}(x)\big) < M_v.$$

Actually, because $m \geqslant r$, an easy computation shows that

$$v\left(\frac{\phi_{t^m}(x)^q}{t^{q-1}}\right) = -q - d(q-1) = -q^{m+1} + q^m - 1 < -q^r = v\big(\big(\phi_{t^m}(x)\big)^{q^r}\big).$$

This shows that $v(\phi_{t^{m+1}}(x)) = -q^{m+1} + q^m - 1 < M_v < 0$ and so, by Lemma 4.12

$$\hat{h}_v(x) = \frac{\hat{h}_v(\phi_{t^{m+1}}(x))}{q^{r(m+1)}} = \frac{q^{m+1} - q^m + 1}{q^{r(m+1)}d} < \frac{q^{m+1}}{q^{m+r}q^{(r-1)m}d} < \frac{q^{1-r}}{d^r},$$

because $d = q^m - 1 < q^m$.

This computation shows that for Drinfeld modules of type

$$\phi_t = \tau^r - t^{1-q}\tau$$

the exponent k from Theorem 4.2 should be at least r. The exact same computation will give us that in the case of a Drinfeld module of the form

$$\phi_t = \tau^r - t^{1-q^{r_0}}\tau^{r_0}$$

for some $1 \leqslant r_0 < r$ and x of valuation $(q^{r_0 m} - 2)$ at a place v that is totally ramified above the place of t with ramification index $q^{r_0 m} - 1$, the exponent k in Theorem 4.2 should be at least $\frac{r}{r_0}$. In Theorem 5.15 we will prove that for nonwildly ramified extensions above places from S_0, we get exponent $k = \frac{r}{r_0}$. But before doing this, we observe that the present example is just a counterexample to Statement 4.1, not to Conjecture 1.1. In other words, the global Lehmer inequality holds for our example even if the local one fails.

Indeed, because x was chosen to have positive valuation at the only place from S, then there exists another place, call it v' which is not in S, for which $v'(x) < 0$. But then by Lemma 4.13, we get that $\hat{h}_{v'}(x) \geqslant \frac{1}{d}$, which means that also $\hat{h}(x) \geqslant \frac{1}{d}$. Thus we obtain a lower bound for the global height as conjectured in Conjecture 1.1.

Now, in order to get to the result of Theorem 5.15 we prove a lemma.

Lemma 5.14. *With the notation from the proof of Theorem* 4.2*, let*

$$l = \mathrm{lcm}_{i\in\{1,\ldots,r-r_0\}}\{q^i - 1\}.$$

If v is not wildly ramified above v_0 (i.e., p does not divide $e(v|v_0)$), then $e(v|v_0)$ divides $l \cdot \alpha$ for every $\alpha \in P_v(n)$ and every $n \geqslant 0$.

Proof. Indeed, from its definition (15), P_v contains $\{0\}$ and integer numbers of the form

$$\alpha := \frac{v(a_i) - v(a_j)}{q^j - q^i} = \frac{e(v|v_0)\cdot(v_0(a_i) - v_0(a_j))}{q^i(q^{j-i} - 1)}, \tag{59}$$

for $j > i$. Hence

$$l \cdot \alpha = \frac{l}{q^{j-i} - 1} \cdot \frac{e(v|v_0)\cdot(v_0(a_i) - v_0(a_j))}{q^i}. \tag{60}$$

By definition,

$$\frac{l}{q^{j-i} - 1} \in \mathbb{Z} \tag{61}$$

because $1 \leqslant j - i \leqslant r - r_0$. Because by our assumption $e(v|v_0)$ is coprime with q^i, and because $\alpha \in \mathbb{Z}$, we conclude from (59) that

$$\frac{v_0(a_i) - v_0(a_j)}{q^i} \in \mathbb{Z}. \tag{62}$$

Using (61) and (62) in (60), we conclude $e(v|v_0)$ divides $l\alpha$.

The set $P_v(1)$ contains integer numbers of the form

$$\alpha_1 := \frac{\alpha - v(a_i)}{q^i}, \tag{63}$$

where $\alpha \in P_v = P_v(0)$ and $a_i \neq 0$. Using that p does not divide $e(v|v_0)$ and that $e(v|v_0) \mid l\alpha$ we get that $e(v|v_0) \mid l\alpha_1$ for all $\alpha_1 \in P_v(1)$. Repeating the process from (63) we obtain all the elements of $P_v(n)$ for every $n \geqslant 1$ and, by induction on n, we conclude that $e(v|v_0) \mid l\alpha_n$ for all $\alpha_n \in P_v(n)$. □

Theorem 5.15. *Let K be a field of characteristic p. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0.*

Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. Let $t \in A$ such that $\phi_t = \sum_{i=r_0}^{r} a_i \tau^i$ is inseparable and assume $a_{r_0} \neq 0$. Let $x \in K^{\mathrm{alg}}$ and let $v \in M_{K(x)}$ be a place lying over v_0. Assume that $\hat{h}_v(x) > 0$.

There exists a constant $C > 0$ depending only on ϕ such that if v is not wildly ramified above v_0, then $\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$.

Proof. Just as we observed in Remark 5.10, it suffices to prove Theorem 5.15 under the hypothesis that ϕ_t is monic in τ.

Let now $d = [K(x) : K]$. We observe again that from Lemma 4.13 it follows that if $v \notin S$ then $\hat{h}_v(x) \geqslant d(v) \geqslant \frac{d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$.

So, from now on we consider the case $v \in S$. First we obtain a sharper result than Lemma 5.6 under our hypothesis that $v|v_0$ is not wildly ramified.

Let $i_0 \geqslant r_0$ be as in Lemma 5.6, i.e. such that $v(\phi_t(x)) = v(a_{i_0} x^{q^{i_0}})$. Then, using the result of Lemma 5.14 (with the notation for l as in Lemma 5.14) in (29) we see that

$$v(x) + \frac{v(a_{i_0})}{q^{i_0} - 1} \leqslant -\frac{\frac{e(v|v_0)}{l}}{q^{i_0} - 1}, \tag{64}$$

if $v(x) \in P_v(n)$ for some n. Then also (46) changes into

$$y_m \leqslant \frac{1}{q^{i_0} - 1}\left(-q^{i_0 m}\frac{e(v|v_0)}{l} - v(a_{i_0})\right). \tag{65}$$

So, then we choose m' minimal such that

$$q^{r_0 m'} \geqslant cl, \tag{66}$$

where $c = c_{v_0}$ is the same as in (47). Thus m' depends only on v_0, ϕ and K, but the dependence on v_0 and K can be considered as part of the dependence on ϕ (see Remark 5.10). We redo the computations from (51) to (55), this time with m' in place of m and because of (65) and (66), we get that

$$\hat{h}_v(x) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}} \quad \text{or} \quad \bigl(v(x), \mathrm{ac}_{\pi_v}(x)\bigr) \in P_v(n) \times R_{v,n}\bigl(v(x)\bigr) \quad \text{for some } n \leqslant m'. \tag{67}$$

At this moment we can redo the argument from the proof of Theorem 4.2 using $\bigcup_{0\leqslant n\leqslant m'} P_v(n)$ instead of $\bigcup_{0\leqslant n\leqslant m} P_v(n)$, only that now $z' := \sum_{n=0}^{m'} |P_v(n)|$ is independent not only of x, but also of $e(v|v_0)$ (as m' depends only on ϕ). We conclude once again that there exists b, a polynomial in t of degree at most $z'f$ such that

$$\hat{h}_v\bigl(\phi_b(x)\bigr) > \frac{c_1 d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}.$$

But because both f and z' depend only on ϕ, we conclude that indeed,

$$\hat{h}(x) \geqslant \frac{C d(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$$

with $C > 0$ depending only on ϕ. $\square$

6. The generic characteristic case

We continue with our notation from Section 4. The following example shows that we cannot expect Theorem 1.2 to be valid in full generality in the case the Drinfeld module has generic characteristic.

Example 6.1. We discuss now Statement 4.1 for Drinfeld modules of generic characteristic. Consider the Carlitz module defined on $\mathbb{F}_p[t]$ (for $p > 2$) by $\phi_t = t\tau^0 + \tau$, where $\tau(x) = x^p$ for all x. Take $K = \mathbb{F}_p(t)$. Let L be a finite extension of K which is totally ramified above ∞ and let the ramification index equal $d = [L : K]$. Also, let v be the unique valuation of L sitting above ∞.

Let $x \in L$ be of valuation nd at v for some $n \geqslant 1$. An easy computation shows that for all $m \in \{1, \dots, n\}$, $v(\phi_{t^m}(x)) = dn - dm$. So, in particular $v(\phi_{t^n}(x)) = 0$ and so,

$$v\bigl(\phi_{t^{n+1}}(x)\bigr) = -d < M_v = \frac{-d}{p-1} \quad \text{because } p > 2.$$

This shows, after using Lemma 4.12, that $\hat{h}_v(\phi_{t^{n+1}}(x)) = \frac{d}{d} = 1$. This in turn implies that

$$\hat{h}_v(x) = \frac{1}{p^{n+1}}.$$

But we can take n arbitrarily large, which shows that there is no way to obtain a result similar to Theorem 4.2 for generic characteristic Drinfeld modules.

We can easily modify the above construction so that it also works when $p = 2$.

The reader might recognize in this example the analytic uniformization at the place ∞ for ϕ, present in Tamagawa's proof for the rigidity of Drinfeld modules of generic characteristic (see the proof of Theorem 4.13.9 from [9]). The idea is that over any function field, there are points arbitrary close to 0 in the ∞-adic topology, which have arbitrary small positive local height at ∞.

The next theorem shows that Example 6.1 is in some sense the only way Theorem 4.2 fails for Drinfeld modules of generic characteristic.

Theorem 6.2. *Let K be a field of characteristic p. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0.*

Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Let $x \in K^{\mathrm{alg}}$ and let v be a place of $K(x)$ that lies over v_0. Assume $\hat{h}_v(x) > 0$.

If v_0 does not lie over the place ∞ of Frac(A)*, then there exist two positive constants C and k depending only on ϕ (and K) such that $\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{k-1}}$.*

Proof. Let $t \in A$ be a nonconstant element and $\phi_t = t\tau^0 + \sum_{i=r_0}^{r} a_i\tau^i$, where a_{r_0} and a_r are nonzero (and $1 \leqslant r_0 \leqslant r$).

Again, as we mentioned before, it suffices to prove this theorem under the hypothesis that ϕ_t is monic in τ. Also, if $v \notin S$, Theorem 6.2 holds as shown by Lemma 4.13.

The analysis of local heights from Section 4 applies to both finite and generic characteristic. So, we still get the conclusion of Lemma 4.17. Thus, if $v(x) \leqslant 0$ then either $\hat{h}_v(x) \geqslant \frac{-d(v)M_{v_0}e(v|v_0)}{q^r}$ or $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$, with $|P_v|$ and $|R_v(v(x))|$ depending only on q and r (the upper bounds for their cardinalities are slightly larger than in the case of a Drinfeld module of finite characteristic, because the maximal number of segments in the Newton polygon for ϕ_t is r and not $r - 1$).

We know from our hypothesis (v does not lie over ∞) that $v(t) \geqslant 0$ and so,

$$v(tx) \geqslant v(x). \tag{68}$$

Now, if $v(x) \geqslant N_v$ (with N_v defined as in Lemma 5.1), then $v(a_i x^{q^i}) \geqslant v(x)$, for all $i \geqslant r_0$ (by the definition of N_v) and using also Eq. (68), we get

$$v\big(\phi_t(x)\big) \geqslant v(x) \geqslant N_v.$$

Iterating this computation we get that $v(\phi_{t^n}(x)) \geqslant N_v$, for all $n \geqslant 1$ and so, $\hat{h}_v(x) = 0$, contradicting the hypothesis of our theorem. This argument is the equivalent of Lemma 5.1 for Drinfeld modules of generic characteristic under the hypothesis $v(t) \geqslant 0$.

Thus it must be that $v(x) < N_v$. Then Lemma 5.2 holds identically. This yields that either $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ or $v(\phi_t(x)) < v(x)$.

From this point on, the proof continues just as for Theorem 4.2. We form just as before the sets $P_v(n)$. We conclude once again as in (54) that *either*

$$\hat{h}_v(x) \geqslant \frac{-M_{v_0}d(v)}{q^{2r}c^{\frac{r}{r_0}}e(v|v_0)^{\frac{r}{r_0}-1}}$$

with the same $c > 0$ depending only on ϕ (and v_0) as in the proof of Theorem 4.2, *or*

$$v(x) \in P_v(n) \quad \text{and} \quad \mathrm{ac}_{\pi_v}(x) \in R_{v,n}\big(v(x)\big) \quad \text{for some } n \leqslant m,$$

where m is of the order of $\log e(v|v_0)$. We observe that when we use Eqs. (42), (44), (45), (46) the index i_0 is still at least $r_0 \geqslant 1$. This is the case because if $v(x) < N_v$ and $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then there exists $i_0 \geqslant 0$ such that

$$v\big(\phi_t(x)\big) = v(a_{i_0}) + q^{i_0} v(x) = \min_{i \in \{0\} \cup \{r_0, \dots, r\}} v\big(a_i x^{q^i}\big).$$

But $v(x) < N_v$ means that there exists at least one index $i \in \{r_0, \dots, r\}$ such that

$$v(tx) \geqslant v(x) > v\big(a_i x^{q^i}\big).$$

Finally, Lemma 5.9 finishes the proof of Theorem 6.2. □

So, we get the conclusion for Theorem 6.2 in the same way as in the proof of Theorem 4.2. The difference made by v not lying above ∞ is that for $v(x) \geqslant 0$, $v(\phi_t(x))$ can decrease only if $v(x) < N_v$, i.e. only if there exists $i \geqslant 1$ such that $v(a_i x^{q^i}) < v(x)$. If v lies over ∞, then $v(tx) < v(x)$ and so, $v(\phi_t(x))$ might decrease just because of the $t\tau^0$ term from ϕ_t. Thus, in that case, as Example 6.1 showed, we can start with x having arbitrarily large valuation and we are able to decrease it by applying ϕ_t to it repeatedly, making the valuation of $\phi_{t^n}(x)$ be less than M_v, which would mean that $\hat{h}_v(x) > 0$. But in doing this we will need a number n of steps (of applying ϕ_t) that we will not be able to control; so $\hat{h}_v(x)$ will be arbitrarily small.

It is easy to see that Remarks 5.10, 5.11 and 5.12 are valid also for Theorem 6.2 in the hypothesis that v does not lie over the place ∞ of $\mathrm{Frac}(A)$. Also, just as we were able to derive Theorem 5.15 from the proof of Theorem 4.2, we can do the same thing in Theorem 6.2 and find a specific value of the constant k that will work in the case that v is not wildly ramified above $v_0 \in M_K$. The result is the following theorem whose proof goes along the same lines as the proof of Theorem 5.15.

Theorem 6.3. *Let K be a field of characteristic p. Let $v_0 \in M_K$ and let $d(v_0)$ be the degree of v_0. Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic and let $\phi_t = t\tau^0 + \sum_{i=r_0}^{r} a_i \tau^i$, with $a_{r_0} \neq 0$ (of course, $r_0 \geqslant 1$). Assume v_0 does not lie over the place ∞ of* $\mathrm{Frac}(A)$. *There exists a constant $C > 0$, depending only on ϕ such that for every $x \in K^{\mathrm{alg}}$ and every place $v \in M_{K(x)}$ such that $v|v_0$ and v is not wildly ramified above v_0, if $\hat{h}_v(x) > 0$ then $\hat{h}_v(x) \geqslant \frac{Cd(v)}{e(v|v_0)^{\frac{r}{r_0}-1}}$.*

We can also construct an example similar to (5.13) which shows that constant $k = \frac{r}{r_0}$ in the above theorem is optimal. Indeed, if we take a Drinfeld module ϕ defined on $\mathbb{F}_q[t]$ by

$$\phi_t = t\tau^0 + t^{1-q^{r_0}}\tau^{r_0} + \tau^r$$

and x as in Example 5.13 then a similar computation will show that we cannot hope for an exponent k smaller than $\frac{r}{r_0}$.

The constants C in Theorems 5.15, 6.2 and 6.3 and the constant k in Theorem 6.2 have the same dependency on q, r and ϕ as explained in the proof of Theorem 4.2.

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